Sequences & Series Practice Sheet (Student Copy)

Praveen Chhikara

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Let $a_1 = 1$ and $a_n = a_{n-1} + 4,\ n \geq 2$. Then,

$$\lim_{n \to \infty} \left[\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \cdots + \frac{1}{a_{n-1} a_n} \right]$$

is equal to ________________

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Let $S_n = \sum_{k=1}^{n} \frac{1}{k}$ and $I_n = \int_1^n \frac{x - [x]}{x^2}\, dx$. Then, $S_{10} + I_{10}$ is equal to

(A) $\ln 10 + 1$

(B) $\ln 10 - 1$

(C) $\ln 10 - \frac{1}{10}$

(D) $\ln 10 + \frac{1}{10}$

3

Consider the power series $\sum_{n=0}^{\infty} a_n z^n$, where $a_n = \begin{cases} \frac{1}{3^n} & \text{if } n \text{ is even} \\ \frac{1}{5^n} & \text{if } n \text{ is odd.} \end{cases}$

The radius of convergence of the series is equal to ___________

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The radius of convergence of the power series $\sum_{n=0}^{\infty} 4^{(-1)^n n} z^{2n}$ is ____________

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Let$x_0 = 0$. Define $x_{n+1} = \cos x_n$ for every $n \geq 0$. Then

(A) $\{x_n\}$ is increasing and convergent
(B) $\{x_n\}$ is decreasing and convergent
(C) $\{x_n\}$ is convergent and $x_{2n} < \lim_{m\to\infty} x_m < x_{2n+1}$ for every $n \in \mathbb{N}$
(D) $\{x_n\}$ is not convergent

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Let $\{a_n\}$ be the sequence of consecutive positive solutions of the equation $\tan x = x$ and let $\{b_n\}$ be the sequence of consecutive positive solutions of the equationtan $\sqrt{x} = x$. Then

(A) $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges but $\sum_{n=1}^{\infty} \frac{1}{b_n}$ diverges

(B) $\sum_{n=1}^{\infty} \frac{1}{a_n}$ diverges but $\sum_{n=1}^{\infty} \frac{1}{b_n}$ converges

(C) Both $\sum_{n=1}^{\infty} \frac{1}{a_n}$ and $\sum_{n=1}^{\infty} \frac{1}{b_n}$ converge

(D) Both $\sum_{n=1}^{\infty} \frac{1}{a_n}$ and $\sum_{n=1}^{\infty} \frac{1}{b_n}$ diverge

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The value of the limit

$$\lim_{n\to\infty} \frac{2^{-n^2}}{\sum_{k=n+1}^{\infty} 2^{-k^2}}$$

is

(A) 0 (B) some $c \in (0,1)$ (C) 1 (D) ∞

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Let $S = \left\{x \in \mathbb{R} : x \geq 0,\ \sum_{n=1}^{\infty} x^{\sqrt{n}} < \infty\right\}$. Then the supremum of S is

(A) 1 (B) $\frac{1}{e}$ (C) 0 (D) ∞

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The series $\sum_{m=1}^{\infty} x^{\ln m}$, $x > 0$, is convergent on the interval

(A) $(0, 1/e)$ (B) $(1/e, e)$ (C) $(0, e)$ (D) $(1, e)$

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Let $x = (x_1, x_2, \ldots) \in l^4$, $x \neq 0$. For which one of the following values of p, the series $\sum_{i=1}^{\infty} x_i y_i$ converges for every $y = (y_1, y_2, \ldots) \in l^p$?

(A) 1 (B) 2 (C) 3 (D) 4

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Consider the power series $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$ and $\sum_{n=1}^{\infty} \frac{x^n}{n}$. Then

(A) both converge on $(-1,1]$

(B) both converge on $[-1,1)$

(C) exactly one of them converges on $(-1,1]$

(D) none of them converges on $[-1,1)$

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Which one of the following statements holds?

(A) The series $\sum_{n=0}^{\infty} x^n$ converges for each $x \in [-1,1]$

(B) The series $\sum_{n=0}^{\infty} x^n$ converges uniformly in $(-1,1)$

(C) The series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges for each $x \in [-1,1]$

(D) The series $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ converges uniformly in $(-1,1)$

13

For a sequence $\{a_n\}$ of real numbers, which of the following is a negation of the statement '$\lim_{n\to\infty} a_n = 0$'?

(a) There exists $\varepsilon > 0$ such that the set $\{n \in \mathbb{N} \mid |a_n| > \varepsilon\}$ is infinite.

(b) For any $M > 0$, there exists $N \in \mathbb{N}$ such that $|a_n| > M$ for all $n \geq N$.

(c) There exists a nonzero real number a such that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ with $|a_n - a| < \varepsilon$ for all $n \geq N$.

(d) For any $a \in \mathbb{R}$, and every $\varepsilon > 0$, there exist infinitely many n such that $|a_n - a| > \varepsilon$.

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Which of the following is false ?

A. $\sum_{n=1}^{\infty} \sin\frac{1}{n}$ diverges

B. $\sum_{n=1}^{\infty} \sin\frac{1}{n^2}$ converges

C. $\sum_{n=1}^{\infty} \cos\frac{1}{n}$ diverges

D. $\sum_{n=1}^{\infty} \cos\frac{1}{n^2}$ converges

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The value of the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$ is

A. 1
B. 2
C. 3
D. 4.

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Let $A = \{\sum_{i=1}^{\infty} \frac{a_i}{5^i} : a_i = 0, 1, 2, 3 \text{ or } 4\} \subset \mathbb{R}$. Then

A. A is a finite set
B. A is countably infinite
C. A is uncountable but does not contain an open interval
D. A contains an open interval.

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Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of real numbers such that the series $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ converge. Then the series $\sum_{n=1}^{\infty} a_n b_n$

A. is absolutely convergent
B. may not converge
C. is always convergent, but may not converge absolutely
D. converges to 0.

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The limit

$$\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{2n}\right)=$$

A. e
B. 2
C. $\log_e 2$
D. e^2.

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Let $\{a_n\}$ be a sequence of real numbers such that $|a_{n+1}-a_n|\leq\frac{n^2}{2^n}$ for all $n\in\mathbb{N}$. Then

A. The sequence $\{a_n\}$ may be unbounded
B. The sequence $\{a_n\}$ is bounded but may not converge
C. The sequence $\{a_n\}$ has exactly two limit points
D. The sequence $\{a_n\}$ is convergent.

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Let $\{a_n\}$ be a sequence of real numbers. Which of the following is true ?

A. If $\sum a_n$ converges, then so does $\sum a_n^4$
B. If $\sum |a_n|$ converges, then so does $\sum a_n^2$
C. If $\sum a_n$ diverges, then so does $\sum a_n^3$
D. If $\sum |a_n|$ diverges, then so does $\sum a_n^2$.

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The series $\sum_{n=1}^{\infty} \frac{\cos(3^n x)}{2^n}$

A. Diverges, for all rational $x \in \mathbb{R}$
B. Diverges, for some irrational $x \in \mathbb{R}$
C. Converges, for some but not all $x \in \mathbb{R}$
D. Converges, for all $x \in \mathbb{R}$.

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A complex number $\alpha \in \mathbb{C}$ is called *algebraic* if there is a non-zero polynomial $P(x) \in \mathbb{Q}[x]$ with rational coefficients such that $P(\alpha) = 0$. Which of the following statements is true ?

A. There are only finitely many algebraic numbers
B. All complex numbers are algebraic
C. $\sin(\frac{\pi}{3}) + \cos(\frac{\pi}{4})$ is algebraic
D. None of the above.

23

How many finite sequences $x_1, x_2, \ldots, x_m$ are there such that each $x_i = 1$ or 2, and $\sum_{i=1}^{m} x_i = 10$?

A. 89
B. 91
C. 92
D. 120.

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Let A, B, C be three subsets of $\mathbb{R}$. The negation of the following statement

For every $\epsilon > 1$, there exists $a \in A$ and $b \in B$ such that for all $c \in C$, $|a-c| < \epsilon$ and $|b-c| > \epsilon$

is

A. there exists $\epsilon \leq 1$, such that for all $a \in A$ and $b \in B$ there exists $c \in C$ such that $|a-c| \geq \epsilon$ or $|b-c| \leq \epsilon$

B. there exists $\epsilon \leq 1$, such that for all $a \in A$ and $b \in B$ there exists $c \in C$ such that $|a-c| \geq \epsilon$ and $|b-c| \leq \epsilon$

C. there exists $\epsilon > 1$, such that for all $a \in A$ and $b \in B$ there exists $c \in C$ such that $|a-c| \geq \epsilon$ and $|b-c| \leq \epsilon$

D. there exists $\epsilon > 1$, such that for all $a \in A$ and $b \in B$ there exists $c \in C$ such that $|a-c| \geq \epsilon$ or $|b-c| \leq \epsilon$.

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Let $a_n = (n+1)^{100} e^{-\sqrt{n}}$ for $n \geq 1$. Then the sequence $(a_n)_n$ is

A. unbounded
B. bounded but does not converge
C. bounded and converges to 1
D. bounded and converges to 0.

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Consider the sequences

$$x_n = \sum_{j=1}^{n} \frac{1}{j}$$
$$y_n = \sum_{j=1}^{n} \frac{1}{j^2}$$

Then $\{x_n\}$ is Cauchy but $\{y_n\}$ is not.

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Let $x_1 \in (0,1)$ be a real number between 0 and 1. For $n > 1$, define

$$x_{n+1} = x_n - x_n^{n+1}.$$

Then $\lim_{n\to\infty} x_n$ exists.

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Suppose $\{a_i\}$ is a sequence in $\mathbb{R}$ such that $\sum |a_i||x_i| < \infty$ whenever $\sum |x_i| < \infty$. Then $\{a_i\}$ is a bounded sequence.

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Consider the function $f(x) = ax + b$ with $a, b \in \mathbb{R}$. Then the iteration

$$x_{n+1} = f(x_n); \qquad\qquad n \geq 0$$

for a given x_0 converges to $b/(1-a)$ whenever $0 < a < 1$.

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The inequality

$$\sqrt{n+1} - \sqrt{n} < \frac{1}{\sqrt{n}}$$

is false for all n such that $101 \leq n \leq 2000$.

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$\lim_{n\to\infty} (n+1)^{1/3} - n^{1/3} = \infty.$

32

Let S be the set of all sequences $\{a_1, a_2, ..., a_n, ...\}$ where each entry a_i is either 0 or 1. Then S is countable.

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Let $\{a_n\}$ be any non-constant sequence in $\mathbb{R}$ such that $a_{n+1} = \frac{a_n + a_{n+2}}{2}$ for all $n \geq 1$. Then $\{a_n\}$ is unbounded.

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The series

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$$

is divergent.

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The inequality $\sum_{n=0}^{\infty} \frac{(\log\log 2)^n}{n!} > \frac{3}{5}$ holds.

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Consider the sequence $\{x_n\}$ defined by $x_n = \frac{[nx]}{n}$ for $x \in \mathbb{R}$ where $[\cdot]$ denotes the integer part. Then $\{x_n\}$

(a) converges to x.

(b) converges but not to x.

(c) does not converge

(d) oscillates

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The function $f_n(x) = n \sin(x/n)$

(a) does not converge for any x as $n \to \infty$.

(b) converges to the constant function 1 as $n \to \infty$.

(c) converges to the function x as $n \to \infty$.

(d) does not converge for all x as $n \to \infty$.

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The value of the infinite product

$$\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right)$$

is 1.

39

The series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n}$$

diverges.

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The sum of the series

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\cdots+\frac{1}{100.101}$$

is

(a) $\frac{99}{101}$
(b) $\frac{98}{101}$
(c) $\frac{99}{100}$
(d) None of the above.

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Let f be an one to one function from the closed interval $[0,1]$ to the set of real numbers $\mathbb{R}$, then

(a) f must be onto.
(b) range of f must contain a rational number.
(c) range of f must contain an irrational number.
(d) range of f must contain both rational and irrational numbers.

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The sequence $\sqrt{7}, \sqrt{7+\sqrt{7}}, \sqrt{7+\sqrt{7+\sqrt{7}}}, \cdots$ converges to

(a) $\frac{1+\sqrt{33}}{2}$
(b) $\frac{1+\sqrt{32}}{2}$
(c) $\frac{1+\sqrt{30}}{2}$
(d) $\frac{1+\sqrt{29}}{2}$

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The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$$

(a) converges but not absolutely.

(b) converges absolutely.

(c) diverges.

(d) none of the above.

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Let $u_n = \sin\left(\frac{\pi}{n}\right)$ and consider the series $\sum u_n$. Which of the following statements is false?

(a) $\sum u_n$ is convergent.

(b) $u_n \to 0$ as $n \to \infty$.

(c) $\sum u_n$ is divergent.

(d) $\sum u_n$ is absolutely convergent.

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Define $\{x_n\}$ as $x_1 = 0.1,\ x_2 = 0.101,\ x_3 = 0.101001, \ldots\ldots$ Then the sequence $\{x_n\}$

(a) converges to a rational number.

(b) converges to an irrational number.

(c) does not converge.

(d) oscillates.